EE 232: Lightwave Devices
Lecture #6 – Fermi's Golden Rule

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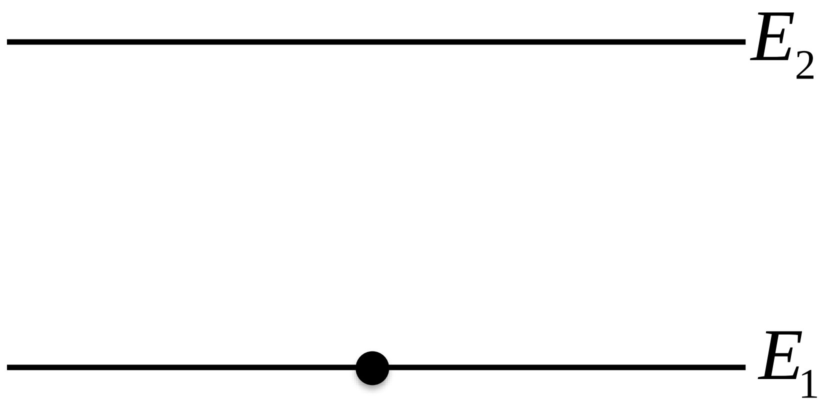
Two-level system

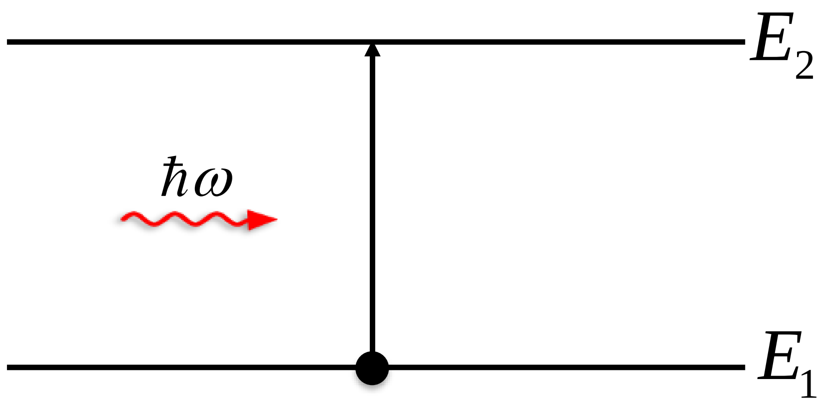


$$i\hbar \frac{\partial}{\partial t}\left|\Psi(t)\right\rangle = \hat{H}_0\left|\Psi(t)\right\rangle$$

$$\left|\Psi(t)\right\rangle = C_1\left|\psi_1\right\rangle e^{-iE_1 t/\hbar} + C_2\left|\psi_2\right\rangle e^{-iE_2 t/\hbar}$$

Where $|C_1|^2$ and $|C_2|^2$ are the probability that an electron is in state $|1\rangle$ or $|2\rangle$ respectively.

Time-dependent perturbation



$$i\hbar\frac{\partial}{\partial t}\left|\Psi(t)\right\rangle = \left[\hat{H}_0 + \hat{H}'(\mathbf{r},t)\right]\left|\Psi(t)\right\rangle$$

$$\left|\Psi(t)\right\rangle = C_1(t)\left|\psi_1\right\rangle e^{-iE_1 t/\hbar} + C_2(t)\left|\psi_2\right\rangle e^{-iE_2 t/\hbar}$$

We presume that the perturbation is small such that the eigenstates are not modified, but the probability of finding an electron in each state may change with time.

Time-dependent perturbation

Plug $\Psi(t)$ into Schrodinger's equation

$$i\hbar\frac{\partial}{\partial t}\left[C_1(t)\left|\psi_1\right\rangle e^{-iE_1t/\hbar}+C_2(t)\left|\psi_2\right\rangle e^{-iE_2t/\hbar}\right]$$

$$=\left[\hat{H}_0+\hat{H}'(\mathbf{r},t)\right]\left[C_1(t)\left|\psi_1\right\rangle e^{-iE_1t/\hbar}+C_2(t)\left|\psi_2\right\rangle e^{-iE_2t/\hbar}\right]$$

After quite a bit of rearranging we find two coupled equations:

$$i\hbar\frac{dC_1(t)}{dt}=\left\langle\psi_1\right|\hat{H}'(\mathbf{r},t)\left|\psi_1\right\rangle C_1(t)+\left\langle\psi_1\right|\hat{H}'(\mathbf{r},t)\left|\psi_2\right\rangle C_2(t)e^{-i\omega_0 t}$$

$$i\hbar\frac{dC_2(t)}{dt}=\left\langle\psi_2\right|\hat{H}'(\mathbf{r},t)\left|\psi_2\right\rangle C_2(t)+\left\langle\psi_2\right|\hat{H}'(\mathbf{r},t)\left|\psi_1\right\rangle C_1(t)e^{i\omega_0 t}$$

where $\omega_0 = E_2 - E_1$

Dipole approximation

The perturbing electric-field from the incident light can be written $\mathbf{E} = \hat{x}\mathrm{E}_0\cos(\omega\mathrm{t}\text{-kz})$

The electric-field will not vary much over the length of our quantum two-level system and therefore we can place our system at the origin and evaluate the field there such that $\mathbf{E} = \hat{x}\mathrm{E}_0\cos(\omega\mathrm{t})$.

dipole approximation

$\mathbf{F} = -q\mathbf{E}$ (force on electron)

$$\text{Potential energy } = -\int \mathbf{F}\cdot d\vec{x}$$

$$= qxE_0\cos(\omega t)$$

This potential energy is precisely equal to our perturbing Hamiltonian in this dipole approximation.

$$\boxed{\hat{H}'(x,t) = qxE_0\cos(\omega t)}$$

Dipole approximation

$$\langle \psi_1 | \hat{H}'(x,t) | \psi_2 \rangle = \int \psi_1^* (qxE_0 \cos(\omega t)) \psi_2 dV$$

$$= qE_0 \cos(\omega t) \mathbf{x}_{12} \quad \text{where } \mathbf{x}_{12} = \langle \psi_1 | x | \psi_2 \rangle$$

$$\langle \psi_1 | \hat{H}'(x,t) | \psi_1 \rangle = \langle \psi_2 | \hat{H}'(x,t) | \psi_2 \rangle = 0 \quad \text{since } \hat{H}'(x,t) \text{ is odd function}$$

Then,

$$\boxed{\begin{aligned} & i\frac{dC_1(t)}{dt} = v\cos(\omega t) C_2(t) e^{-i\omega_0 t} \\ & i\frac{dC_2(t)}{dt} = v\cos(\omega t) C_1(t) e^{i\omega_0 t} \qquad \text{where } \hbar v = qE_0 \mathbf{x}_{12} \end{aligned}}$$

Weak-field limit

Presume the two-level system is in the initial state:

$$C_1(0)=1 \text{ and } C_2(0)=0$$

Further, presume the perturbation is weak and the transition rate from the first to second state is slow. i.e., assume:

$$C_1(t)=1 \text{ and } \frac{dC_1(t)}{dt}=0$$

Plug this guess into the coupled equations on previous slide, to find:

$$C_2(t)=\frac{\nu}{2}\left[\frac{1-e^{i(\omega_0-\omega)t}}{\omega_0-\omega}+\frac{1-e^{i(\omega_0+\omega)t}}{\omega_0+\omega}\right]$$

Weak-field limit

Rewrite

$$C_2(t) = -2i\,\frac{\nu}{2}\left[\frac{e^{i(\omega_0-\omega)t/2}\sin[(\omega_0-\omega)t/2]}{\omega_0-\omega} + \frac{e^{i(\omega_0+\omega)t/2}\sin[(\omega_0+\omega)t/2]}{\omega_0+\omega}\right]$$

$$\left|C_2(t)\right|^2 = C_2(t)C_2^*(t)$$

$$= \left|\frac{\nu}{2}\right|^2\left[\left(\frac{\sin[(\omega_0-\omega)t/2]}{(\omega_0-\omega)/2}\right)^2 + \left(\frac{\sin[(\omega_0+\omega)t/2]}{(\omega_0+\omega)/2}\right)^2 + 2\cos(2\omega t)\,\frac{\sin[(\omega_0-\omega)t/2]}{(\omega_0-\omega)/2}\,\frac{\sin[(\omega_0+\omega)t/2]}{(\omega_0+\omega)/2}\right]$$

$$\boxed{\approx \left|\frac{\nu}{2}\right|^2\left[\left(\frac{\sin[(\omega_0-\omega)t/2]}{(\omega_0-\omega)/2}\right)^2 + \left(\frac{\sin[(\omega_0+\omega)t/2]}{(\omega_0+\omega)/2}\right)^2\right]}$$

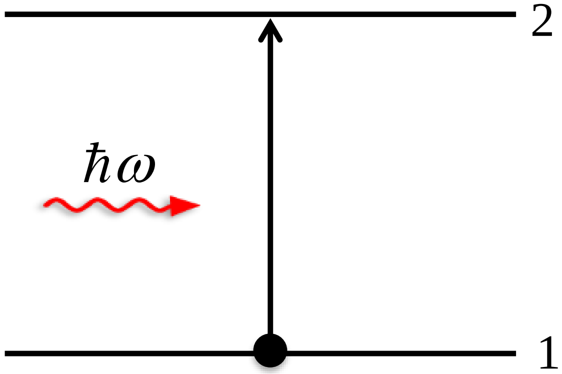
Absorption / stimulated emission

$$\left|C_2(t)\right|^2 = \left|\frac{\nu}{2}\right|^2 \left[\left(\frac{\sin[(\omega_0 - \omega)t/2]}{(\omega_0 - \omega)/2} \right)^2 + \left(\frac{\sin[(\omega_0 + \omega)t/2]}{(\omega_0 + \omega)/2} \right)^2 \right]$$

First term resonant when

$\omega_0 - \omega = 0$

i.e. $E_2 - E_1 - \hbar\omega = 0$

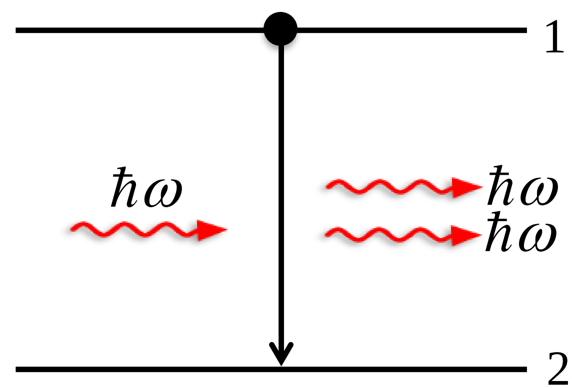


Absorption

Second term resonant when

$\omega_0 + \omega = 0$

i.e. $E_2 - E_1 + \hbar\omega = E_1 - E_2 - \hbar\omega = 0$



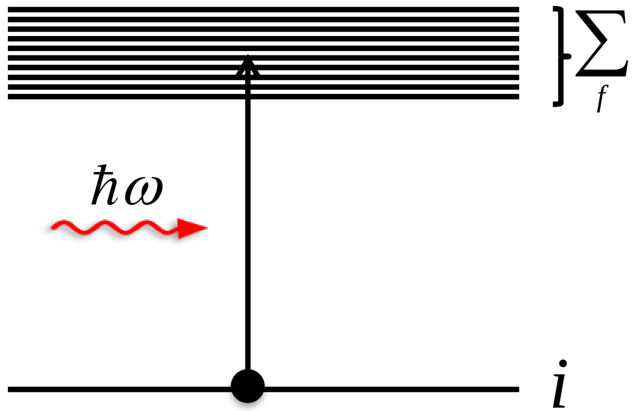
Stimulated emission

Continuum of final states

Consider a transition from the first state to a continuum of final states.
Consider absorption term first. Stimulated emission term has similar derivation

$$P = \sum_f \left| C_f(t) \right|^2$$

(probability to find electron in one of the upper states)



$$= \int \left| C_f(t) \right|^2 g(\omega_{fi}) d\omega_{fi}$$

$g(\omega_{fi})$: density of states

$$= \int \left| \frac{v}{2} \right|^2 \left[\frac{\sin^2[(\omega_{fi} - \omega)t/2]}{[(\omega_{fi} - \omega)/2]^2} \right] g(\omega_{fi}) d\omega_{fi}$$

Assume $g(\omega_{fi})$ is not sharply peaked

and v is independent of ω

$$= \left| \frac{v}{2} \right|^2 g(\omega) \int \frac{\sin^2[(\omega_{fi} - \omega)t/2]}{[(\omega_{fi} - \omega)/2]^2} d\omega_{fi}$$

$$= \left| \frac{v}{2} \right|^2 g(\omega) 2\pi t$$

$$W = \frac{dP}{dt} = \left| \frac{v}{2} \right|^2 g(\omega) 2\pi$$

(transition rate s^{-1})

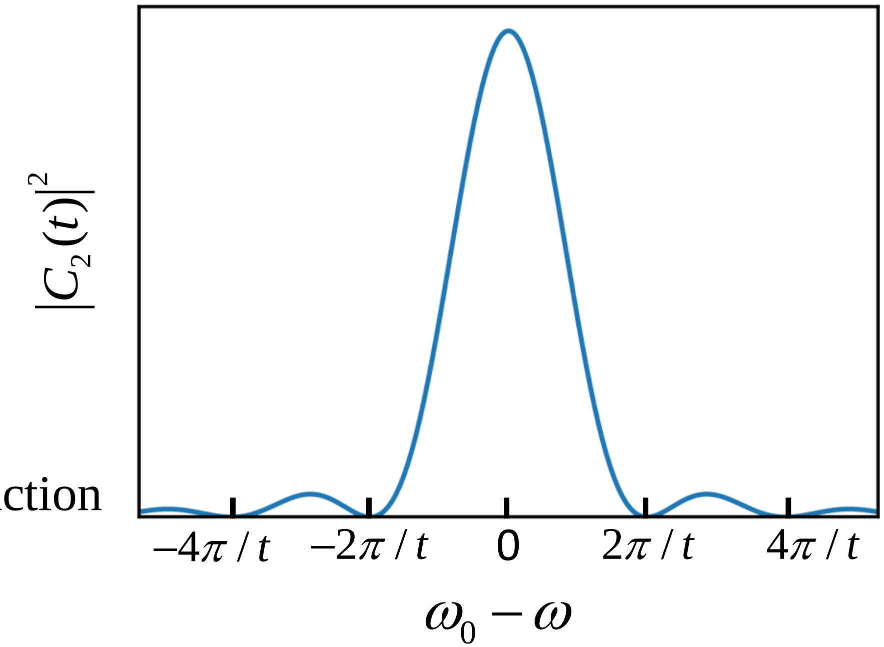
$$= \left| \frac{qE_0 \mathbf{x}_{12}}{2\hbar} \right|^2 g(\omega) 2\pi$$

$$= \frac{2\pi}{\hbar^2} \left| \left\langle \psi_f \left| \frac{qE_0 x}{2} \right| \psi_i \right\rangle \right|^2 g(\omega) = \boxed{\frac{2\pi}{\hbar^2} \left| \hat{H}_{fi} \right|^2 g(\omega)}$$

Alternative derivation

$$|C_2(t)|^2 = \left|\frac{\nu}{2}\right|^2 \left[\left(\frac{\sin[(\omega_0-\omega)t/2]}{(\omega_0-\omega)/2}\right)^2\right]$$

(Absorption)



$$\frac{\sin^2(tx/2)}{x^2} \to \frac{\pi t}{2}\delta(x)$$

$\delta(x)$: Dirac delta function

For long interaction time

$$|C_2(t)|^2 = \left|\frac{\nu}{2}\right|^2 \delta(\omega_{fi}-\omega)2\pi t \qquad w_{fi} = \frac{dP}{dt} = \left|\frac{\nu}{2}\right|^2 \delta(\omega_{fi}-\omega)2\pi$$

$$= \left|\frac{qE_0\mathbf{x}_{12}}{2\hbar}\right|^2 \delta(\omega_{fi}-\omega)2\pi$$

$$= \frac{2\pi}{\hbar^2}\left|\left\langle\psi_f\left|\frac{qE_0x}{2}\right|\psi_i\right\rangle\right|^2 \delta(\omega_{fi}-\omega) = \boxed{\frac{2\pi}{\hbar^2}\left|\hat{H}_{fi}\right|^2 \delta(\omega_{fi}-\omega)}$$

Alternative derivation

Delta function is highly singular unless evaluated in an integral. Again, sum over a continuum of final states.

$$W = \sum_f \frac{2\pi}{\hbar^2} \left| \hat{H}_{fi} \right|^2 \delta(\omega_{fi} - \omega)$$

$$= \int \frac{2\pi}{\hbar^2} \left| \hat{H}_{fi} \right|^2 \delta(\omega_{fi} - \omega) g(\omega_{fi}) d\omega_{fi}$$

$$= \frac{2\pi}{\hbar^2} \left| \hat{H}_{fi} \right|^2 g(\omega)$$

We recover the previous expression for the transition rate

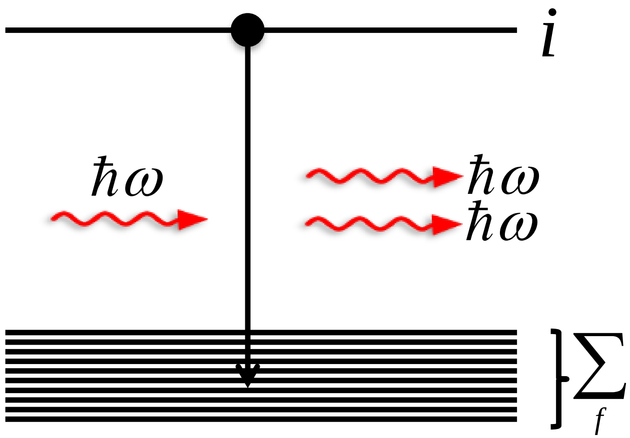
Stimulated emission

Consider the stimulated emission term next.

Follow the same procedure to find the transition rate to a continuum set of final states,

$$W = \frac{2\pi}{\hbar^2} \left| \hat{H}_{fi} \right|^2 g(\omega)$$

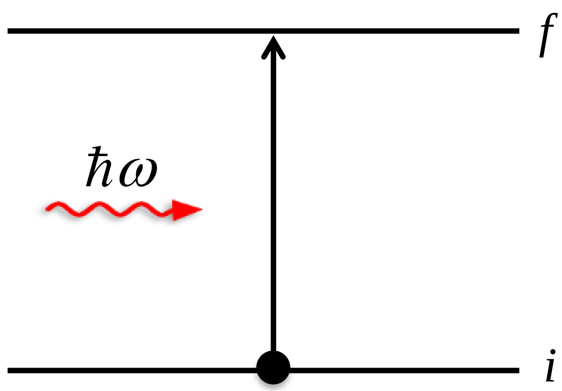
$$w_{if} = \frac{2\pi}{\hbar^2} \left| \hat{H}_{fi} \right|^2 \delta(\omega_{if} - \omega)$$



This form is slightly different than Chuang. It is written this way (using the even property of the delta function) to emphasize that the impinging light energy is positive and the energy of the initial state is higher than the energy of the final state.

Summary: Fermi's Golden Rule

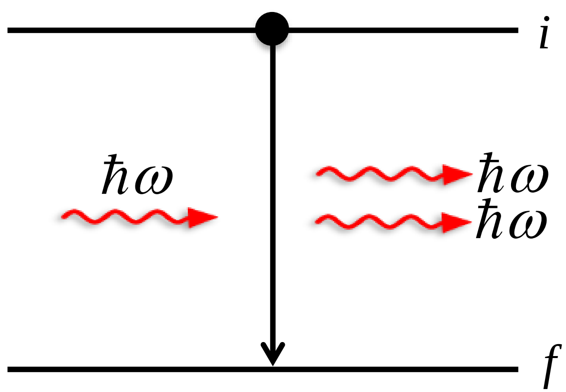
Absorption



$$w_{fi} = \frac{2\pi}{\hbar^2}\left|\hat{H}_{fi}\right|^2 \delta(\omega_{fi} - \omega)$$

$$W = \frac{2\pi}{\hbar^2}\left|\hat{H}_{fi}\right|^2 g(\omega)$$

Stimulated emission



$$w_{fi} = \frac{2\pi}{\hbar^2}\left|\hat{H}_{fi}\right|^2 \delta(\omega_{if} - \omega)$$

$$W = \frac{2\pi}{\hbar^2}\left|\hat{H}_{fi}\right|^2 g(\omega)$$

Fermi's Golden Rule for arbitrary periodic perturbation

We derived Fermi's Golden Rule for a particular periodic perturbation

$$\hat{H}'(x,t) = qxE_0\cos(\omega t) = \frac{qxE_0}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

However, we could have performed the derivation for some arbitrary harmonic perturbation with the form

$$\hat{H}'(\mathbf{r},t) = H'(\mathbf{r})\left(e^{i\omega t} + e^{-i\omega t}\right)$$

Then, for a general harmonic perturbation, the transition rate for absorption is given by:

$$\begin{aligned} w_{fi} &= \frac{2\pi}{\hbar^2}\left|\left\langle \psi_f \middle| \hat{H}'(\mathbf{r}) \middle| \psi_i \right\rangle\right|^2 \delta(\omega_{fi} - \omega) \\ &= \frac{2\pi}{\hbar^2}\left|\hat{H}_{fi}\right|^2 \delta(\omega_{fi} - \omega) \end{aligned}$$